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# The uniqueness theorems of meromorphic functions sharing three values and one pair of values <sup>☆</sup>

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## Abstract

In this paper, we deal with a uniqueness theorem of two meromorphic functions that have three weighted sharing values and one pair of values. The results in this paper improve those given by G.G. Gundersen, G. Brosch, T.C. Alzahary, T.C. Alzahary and H.X. Yi, I. Lahiri and P. Sahoo, and other authors.

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## 1. Introduction and main results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [7]. It will be convenient to let  $E$  denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any nonconstant meromorphic function  $h(z)$ , we denote by  $S(r, h)$  any quantity satisfying

$$S(r, h) = o(T(r, h)) \quad (r \rightarrow \infty, r \notin E).$$

Let  $f(z)$  and  $g(z)$  be two nonconstant meromorphic functions, and let  $a$  be a value in the extended plane. We say that  $f$  and  $g$  share the value  $a$  CM, provided that  $f$  and  $g$  have the same  $a$ -points with the same multiplicities. Similarly, we say that  $f$  and  $g$  share the value  $a$  IM, provided that  $f$  and  $g$  have the same  $a$ -points ignoring multiplicities (see [16]). We say that  $a(z)$  is a small function of  $f$ , if  $a(z)$  is a meromorphic function satisfying  $T(r, a(z)) = o(T(r, f))$  ( $r \notin E$ ), as  $r \rightarrow \infty$ . In addition, we need the following definition.

**Definition 1.1.** (See [1, Definition 1].) Let  $p$  be a positive integer and  $a \in C \cup \{\infty\}$ . Then by  $N_p(r, \frac{1}{f-a})$  we denote the counting function of those zeros of  $f - a$  (counted with proper multiplicities) whose multiplicities are not

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greater than  $p$ , by  $\bar{N}_p(r, \frac{1}{f-a})$  we denote the corresponding reduced counting function (ignoring multiplicities). By  $N_p(r, \frac{1}{f-a})$  we denote the counting function of those zeros of  $f - a$  (counted with proper multiplicities) whose multiplicities are not less than  $p$ , by  $\bar{N}_{(p)}(r, \frac{1}{f-a})$  we denote the corresponding reduced counting function (ignoring multiplicities).

Let  $f(z)$  and  $g(z)$  be two nonconstant meromorphic functions, and let  $a$  be a value in the extended plane. Let  $S$  be a subset of distinct elements in the extended plane. Next we define

$$E_f(S) = \bigcup_{a \in S} \{z: f(z) = a\},$$

where each  $a$ -point of  $f$  with multiplicity  $m$  is repeated  $m$  times in  $E_f(S)$  (see[5]). Similarly, we define

$$\bar{E}_f(S) = \bigcup_{a \in S} \{z: f(z) = a\},$$

where each point in  $\bar{E}_f(\{a\})$  is counted only once. We say that  $f$  and  $g$  share the set  $S$  CM, provided  $E_f(S) = E_g(S)$ . We say that  $f$  and  $g$  share the set  $S$  IM, provided  $\bar{E}_f(S) = \bar{E}_g(S)$ . Let  $k$  be a positive integer, we denote by  $\bar{E}_k(a, f)$  the set of zeros of  $f(z) - a$  with multiplicity  $\leq k$ , and each such zero of  $f(z) - a$  is counted only once (see [2, Definition 3]).

In 1926, R. Nevanlinna proved the following theorem.

**Theorem A.** (See [15].) *If  $f$  and  $g$  are distinct nonconstant meromorphic functions that share four values  $a_1, a_2, a_3$  and  $a_4$  CM, then  $f$  is a Möbius transformation of  $g$ , two of the shared values, say  $a_1$  and  $a_2$ , are Picard values, and the cross ratio  $(a_1, a_2, a_3, a_4) = -1$ .*

In 1979, G.G. Gundersen proved the following theorem, which improved Theorem A.

**Theorem B.** (See [6, Theorem 1].) *Let  $f$  and  $g$  be two distinct nonconstant meromorphic functions such that  $f$  and  $g$  share three values CM and share a fourth value IM, then  $f$  and  $g$  share all four values CM, and hence the conclusion of Theorem A holds.*

In 1989, G. Brosch proved the following theorem, which improved Theorems A and B.

**Theorem C.** (See [4].) *Let  $f$  and  $g$  be two distinct nonconstant meromorphic functions such that  $f$  and  $g$  share  $0, 1$  and  $\infty$  CM, and let  $a$  and  $b$  be two distinct finite complex numbers such that  $a, b \notin \{0, 1\}$ . If  $f - a$  and  $g - b$  share  $0$  IM, then  $f$  is a Möbius transformation of  $g$ .*

Regarding Theorem C, it is natural to ask the following question.

**Question 1.1.** (See [8].) *Is it really impossible to relax in any way the nature of sharing any one of  $0, 1$  and  $\infty$  in Theorem C?*

In this paper, we will deal with Question 1.1. To this end we employ the idea of weighted sharing of values which measures how close a shared value is to being shared IM or to being shared CM. The notion is explained in the following definition.

**Definition 1.2.** (See [9, Definition 4].) *Let  $k$  be a nonnegative integer or infinity. For any  $a \in C \cup \{\infty\}$ , we denote by  $E_k(a, f)$  the set of all  $a$ -points of  $f$ , where an  $a$ -point of multiplicity  $m$  is counted  $m$  times if  $m \leq k$ , and  $k + 1$  times if  $m > k$ . If  $E_k(a, f) = E_k(a, g)$ , we say that  $f, g$  share the value  $a$  with weight  $k$ .*

**Remark 1.1.** Definition 1.2 implies that if  $f, g$  share a value  $a$  with weight  $k$ , then  $z_0$  is a zero of  $f - a$  with multiplicity  $m (\leq k)$  if and only if it is a zero of  $g - a$  with multiplicity  $m (\leq k)$ , and  $z_0$  is a zero of  $f - a$  with

multiplicity  $m (> k)$ , if and only if it is a zero of  $g - a$  with multiplicity  $n (> k)$ , where  $m$  is not necessarily equal to  $n$ . Throughout this paper, we write  $f, g$  share  $(a, k)$  to mean that  $f, g$  share the value  $a$  with weight  $k$ . Clearly, if  $f, g$  share  $(a, k)$ , then  $f, g$  share  $(a, p)$  for all integer  $p, 0 \leq p < k$ . Also we note that  $f, g$  share a value  $a$  IM or CM if and only if  $f, g$  share  $(a, 0)$  or  $(a, \infty)$ , respectively.

Using the idea of weighted sharing, T.C. Alzahary proved the following theorem in 2006, which improved Theorem C.

**Theorem D.** (See [2].) *Let  $f$  and  $g$  be two distinct nonconstant meromorphic functions such that  $f$  and  $g$  share  $(a_1, 1), (a_2, \infty)$  and  $(a_3, \infty)$ , where  $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$ , and let  $a (\neq 0, 1)$  and  $b (\neq 0, 1)$  be two finite complex numbers. Further suppose that  $\bar{E}_2(a, f) \subset \bar{E}_g(\{b\})$ , then*

(I) *If  $f$  is a Möbius transformation of  $g$ , then  $f$  and  $g$  assume one of the following nine relations:*

- (i)  $fg \equiv 1$ , with  $ab = 1$ ;
- (ii)  $f + g \equiv 1$ , with  $a + b = 1$ ;
- (iii)  $f \equiv \frac{g}{g-1}$ , with  $ab = a + b$ ;
- (iv)  $f \equiv ag$ , with  $ab = 1$ ;
- (v)  $f \equiv \frac{ag}{b}$ ;
- (vi)  $f \equiv (1-a)g + a$ , with  $ab = a + b$ ;
- (vii)  $f \equiv \frac{(1-a)g}{1-b} + \frac{b-a}{b-1}$ ;
- (viii)  $f \equiv \frac{ag}{g+a-1}$ ,  $a + b = 1$ ;
- (ix)  $f \equiv \frac{a(b-1)g}{(b-a)g + (a-1)b}$ .

(II) *If  $f$  is not any Möbius transformation of  $g$ , then there exists a nonconstant entire function  $\gamma$ , such that  $f$  and  $g$  are given by one of the following nine expressions:*

- (i)  $f = \frac{e^{3\gamma}-1}{e^\gamma-1}$ ,  $g = \frac{e^{-3\gamma}-1}{e^{-\gamma}-1}$ , with  $a = 3/4$  and  $b = 3$ ;
- (ii)  $f = \frac{e^{3\gamma}-1}{e^{2\gamma}-1}$ ,  $g = \frac{e^{-3\gamma}-1}{e^{-2\gamma}-1}$ , with  $a = -3$  and  $b = 3/2$ ;
- (iii)  $f = \frac{e^\gamma-1}{e^{3\gamma}-1}$ ,  $g = \frac{e^{-\gamma}-1}{e^{-3\gamma}-1}$ , with  $a = 4/3$  and  $b = 1/3$ ;
- (iv)  $f = \frac{e^{2\gamma}-1}{e^{3\gamma}-1}$ ,  $g = \frac{e^{-2\gamma}-1}{e^{-3\gamma}-1}$ , with  $a = -1/3$  and  $b = 2/3$ ;
- (v)  $f = \frac{e^{2\gamma}-1}{e^{-\gamma}-1}$ ,  $g = \frac{e^{-2\gamma}-1}{e^\gamma-1}$ , with  $a = 1/4$  and  $b = -2$ ;
- (vi)  $f = \frac{e^\gamma-1}{e^{-2\gamma}-1}$ ,  $g = \frac{e^{-\gamma}-1}{e^{2\gamma}-1}$ , with  $a = 4$  and  $b = -1/2$ ;
- (vii)  $f = \frac{e^{2\gamma}-1}{\lambda e^\gamma-1}$ ,  $g = \frac{e^{-2\gamma}-1}{\frac{1}{\lambda}e^{-\gamma}-1}$ , with  $\lambda^2 \neq 1$ ,  $a^2\lambda^2 = 4(a-1)$  and  $b = 2$ ;
- (viii)  $f = \frac{e^\gamma-1}{\lambda e^{2\gamma}-1}$ ,  $g = \frac{e^{-\gamma}-1}{\frac{1}{\lambda}e^{-2\gamma}-1}$ , with  $\lambda \neq 1$ ,  $4a(1-a)\lambda = 1$  and  $b = 1/2$ ;
- (ix)  $f = \frac{e^\gamma-1}{\lambda e^{-\gamma}-1}$ ,  $g = \frac{e^{-\gamma}-1}{\frac{1}{\lambda}e^\gamma-1}$ , with  $\lambda \neq 1$ ,  $(1-a)^2 + 4a\lambda = 0$  and  $b = -1$ .

Using the idea of weighted sharing, we will prove the following theorem, which improves Theorem D and deals with Question 1.1.

**Theorem 1.1.** *Let  $f$  and  $g$  be two distinct nonconstant meromorphic functions such that  $f$  and  $g$  share  $(0, k_1)$   $(1, k_2)$  and  $(\infty, k_3)$ , where  $k_1, k_2$  and  $k_3$  are three positive integers satisfying*

$$k_1 k_2 k_3 > k_1 + k_2 + k_3 + 2, \quad (1.1)$$

*and let  $a$  and  $b$  be two distinct finite complex numbers such that  $a, b \notin \{0, 1\}$ . If  $\bar{E}_2(a, f) \subseteq \bar{E}_g(\{b\})$ , then the conclusions of Theorem D still hold.*

From Theorem 1.1 we can get the following two corollaries.

**Corollary 1.1.** *Let  $f$  and  $g$  be two distinct nonconstant meromorphic functions such that  $f$  and  $g$  share  $(0, k_1)$   $(1, k_2)$  and  $(\infty, k_3)$ , where  $k_1, k_2$  and  $k_3$  are three positive integers satisfying (1.1), and let  $a$  and  $b$  be two distinct*

finite complex numbers such that  $b \notin \{-2, -1, -\frac{1}{2}, 0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1, \frac{3}{2}, 2, 3\}$ . If  $\overline{E}_2(a, f) \subseteq \overline{E}_g(\{b\})$ , then  $f$  is a Möbius transformation of  $g$ , and  $f$  and  $g$  assume one of the nine relations (I)(i)–(ix) in Theorem D.

**Corollary 1.2.** Let  $f$  and  $g$  be two distinct nonconstant entire functions such that  $f$  and  $g$  share  $(0, 1)$  and  $(1, m)$ , where  $m (\geq 2)$  is a positive integer, and let  $a$  and  $b$  be two distinct finite complex numbers such that  $a, b \notin \{0, 1\}$ . If  $\overline{E}_f(\{a\}) \subseteq \overline{E}_g(\{b\})$ , then  $f$  and  $g$  assume one of the relations (I)(i), (iii), (xiii), (ix) and (II)(i) and (v) in Theorem D.

From Corollary 1.2 we can deduce the following corollary.

**Corollary 1.3.** Let  $f$  and  $g$  be two distinct nonconstant entire functions such that  $f$  and  $g$  share  $(0, 1)$  and  $(1, m)$ , where  $m (\geq 2)$  is a positive integer, and let  $a$  and  $b$  be two distinct finite complex numbers such that  $a, b \notin \{0, 1\}$ , and such that  $a \notin \{\frac{1}{4}, \frac{3}{4}\}$  or  $b \notin \{-2, 3\}$ . If  $\overline{E}_f(\{a\}) \subseteq \overline{E}_g(\{b\})$ , then  $f$  is a Möbius transformation of  $g$ . Moreover,  $f$  and  $g$  assume one of the relations (I)(i), (iii), (xiii) and (ix) in Theorem D.

Using the idea of weighted sharing, I. Lahiri and P. Sahoo proved the following theorem recently, which improved Theorems C and 1 in [3].

**Theorem E.** (See [10, Theorem 1.1].) Let  $f$  and  $g$  be two distinct nonconstant meromorphic functions such that  $f$  and  $g$  share  $(a_1, 1)$ ,  $(a_2, m)$  and  $(a_3, k)$ , where  $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$ , and  $m$  and  $k$  are two positive integers satisfying  $(m-1)(mk-1) > (1+m)^2$ , and let  $a$  and  $b$  be two distinct finite complex numbers such that  $a, b \notin \{0, 1\}$ . If  $f-a$  and  $g-b$  share 0 IM, then  $f$  and  $g$  share 0, 1 and  $\infty$  CM, and  $f-a$  and  $g-b$  share 0 CM. Moreover,  $f$  and  $g$  assume one of the nine relations (I)(i)–(ix) in Theorem D.

From Theorem 1.1 we can deduce the following result, which improves Theorem E.

**Theorem 1.2.** Let  $f$  and  $g$  be two distinct nonconstant meromorphic functions such that  $f$  and  $g$  share  $(0, k_1)$   $(1, k_2)$  and  $(\infty, k_3)$ , where  $k_1, k_2$  and  $k_3$  are three positive integers satisfying (1.1), and let  $a$  and  $b$  be two distinct finite complex numbers such that  $a, b \notin \{0, 1\}$ . If  $f-a$  and  $g-b$  share 0 IM, then  $f$  is a Möbius transformation of  $g$ . Moreover,  $f$  and  $g$  assume one of the nine relations (I)(i)–(ix) in Theorem D.

## 2. Some lemmas

**Lemma 2.1.** (See [11, Lemma 6].) Let  $f$  and  $g$  be two distinct nonconstant meromorphic functions such that  $f$  and  $g$  share 0, 1 and  $\infty$  IM. If  $f$  is a Möbius transformation of  $g$ , then  $f$  and  $g$  satisfy one of the following relations:

- (i)  $f \cdot g \equiv 1$ ;                      (ii)  $(f-1)(g-1) \equiv 1$ ;
- (iii)  $f+g \equiv 1$ ;                      (iv)  $f \equiv cg$ ;
- (v)  $f-1 \equiv c(g-1)$ ;              (vi)  $[(c-1)f+1] \cdot [(c-1)g-c] \equiv -c$ ;

where  $c (\neq 0, 1)$  is a finite complex number.

**Lemma 2.2.** (See [17, Lemma 2.6].) Let  $f$  and  $g$  be two distinct nonconstant meromorphic functions such that  $f$  and  $g$  share  $(0, k_1)$ ,  $(1, k_2)$  and  $(\infty, k_3)$ , where  $k_1, k_2$  and  $k_3$  are three positive integers satisfying (1.1), then  $\overline{N}_{(2)}(r, \frac{1}{f}) + \overline{N}_{(2)}(r, \frac{1}{f-1}) + \overline{N}_{(2)}(r, f) = S(r, f)$ .

**Lemma 2.3.** (See [12, Theorem 4.2].) Let  $f$  and  $g$  be two distinct nonconstant meromorphic functions sharing 0, 1 and  $\infty$  CM. If there exists a finite complex number  $a (\neq 0, 1)$  such that  $a$  is not a Picard value of  $f$ , and such that

$$N_1\left(r, \frac{1}{f-a}\right) \leq uT(r, f) + S(r, f),$$

where  $u < \frac{1}{3}$ , then  $N_1(r, \frac{1}{f-a}) = 0$ , and  $f$  and  $g$  are given by one of the following nine expressions:

- (i)  $f = \frac{e^{3\gamma}-1}{e^\gamma-1}$ ,  $g = \frac{e^{-3\gamma}-1}{e^{-\gamma}-1}$ , with  $a = \frac{3}{4}$ ;      (ii)  $f = \frac{e^{3\gamma}-1}{e^{2\gamma}-1}$ ,  $g = \frac{e^{-3\gamma}-1}{e^{-2\gamma}-1}$ , with  $a = -3$ ;  
 (iii)  $f = \frac{e^\gamma-1}{e^{3\gamma}-1}$ ,  $g = \frac{e^{-\gamma}-1}{e^{-3\gamma}-1}$ , with  $a = \frac{4}{3}$ ;      (iv)  $f = \frac{e^{2\gamma}-1}{e^{3\gamma}-1}$ ,  $g = \frac{e^{-2\gamma}-1}{e^{-3\gamma}-1}$ , with  $a = -\frac{1}{3}$ ;  
 (v)  $f = \frac{e^{2\gamma}-1}{e^{-\gamma}-1}$ ,  $g = \frac{e^{-2\gamma}-1}{e^\gamma-1}$ , with  $a = \frac{1}{4}$ ;      (vi)  $f = \frac{e^\gamma-1}{e^{-2\gamma}-1}$ ,  $g = \frac{e^{-\gamma}-1}{e^{2\gamma}-1}$ , with  $a = 4$ ;  
 (vii)  $f = \frac{e^{2\gamma}-1}{\lambda e^\gamma-1}$ ,  $g = \frac{e^{-2\gamma}-1}{\frac{1}{\lambda}e^{-\gamma}-1}$ , with  $\lambda^2 \neq 1$  and  $a^2\lambda^2 = 4(a-1)$ ;  
 (viii)  $f = \frac{e^\gamma-1}{\lambda e^{2\gamma}-1}$ ,  $g = \frac{e^{-\gamma}-1}{\frac{1}{\lambda}e^{-2\gamma}-1}$ , with  $\lambda \neq 1$  and  $4a(1-a)\lambda = 1$ ;  
 (ix)  $f = \frac{e^\gamma-1}{\lambda e^{-\gamma}-1}$ ,  $g = \frac{e^{-\gamma}-1}{\frac{1}{\lambda}e^\gamma-1}$ , with  $\lambda \neq 1$  and  $(1-a)^2 + 4a\lambda = 0$ ;

where  $\gamma$  is a nonconstant entire function.

**Lemma 2.4.** (See [14, Theorem 1.1].) Let  $f$  and  $g$  be two distinct nonconstant meromorphic functions sharing  $(a_1, k_1)$ ,  $(a_2, k_2)$  and  $(a_3, k_3)$ , where  $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$ , and  $k_1, k_2, k_3$  are three positive integers satisfying (1.1). If  $f$  is not any Möbius transformation of  $g$ , then

- (i)  $N_0(r, \frac{1}{f}) = \bar{N}_0(r, \frac{1}{f}) + S(r, f)$ ,  $\bar{N}(r, \frac{1}{f}) = \bar{N}_0(r, \frac{1}{f}) + S(r, f)$ , the same identities hold for  $g$ ;  
 (ii)  $N_3(r, \frac{1}{f-a}) = S(r, f)$ ,  $N_3(r, \frac{1}{g-a}) = S(r, f)$ ;  
 (iii)  $T(r, f) = \bar{N}(r, \frac{1}{g}) + N_0(r) + S(r, f)$ ,  $T(r, g) = \bar{N}(r, \frac{1}{f}) + N_0(r) + S(r, f)$ ,  $N_0(r) = \bar{N}_0(r) + S(r, f)$ ;  
 (iv)  $T(r, f) = N(r, \frac{1}{f-a}) + S(r, f)$ ,  $T(r, g) = N(r, \frac{1}{g-a}) + S(r, f)$ ;  
 (v)  $T(r, f) + T(r, g) = \bar{N}(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{f-1}) + \bar{N}(r, f) + N_0(r) + S(r, f)$ ;  
 (vi)  $N(r, \frac{1}{f-g}) = \bar{N}(r, \frac{1}{f-g}) + S(r, f)$ ;

where, and in the sequel,  $N_0(r, \frac{1}{f})(\bar{N}_0(r, \frac{1}{f}))$  denotes the counting function corresponding to the zeros of  $f'$  that are not zeros of  $f$  and  $f-1$  (ignoring multiplicities) and  $N_0(r)(\bar{N}_0(r))$  is the counting function of the zeros of  $f-g$  that are not zeros of  $g, g-1$  and  $1/g$  (ignoring multiplicities), and  $a (\neq 0, 1)$  is an arbitrary finite complex number.

**Lemma 2.5.** (See [18, Lemma 6].) Let  $f_1$  and  $f_2$  be two distinct nonconstant meromorphic functions satisfying  $\bar{N}(r, f_j) + \bar{N}(r, 1/f_j) = S(r)$  ( $j = 1, 2$ ). Then either  $\bar{N}_0(r, 1; f_1, f_2) = S(r)$  or there exist two integers  $s, t$  ( $|s| + |t| > 0$ ) such that  $f_1^s f_2^t \equiv 1$ , where, and in the sequel,  $\bar{N}_0(r, 1; f_1, f_2)$  denotes the reduced counting function of  $f_1$  and  $f_2$  related to the common 1-points, and  $T(r) = T(r, f_1) + T(r, f_2)$ ,  $S(r) = o(T(r))$  ( $r \rightarrow \infty, r \notin E$ ) only depending on  $f_1$  and  $f_2$ .

The following result improves Theorem 1.1 in [2].

**Lemma 2.6.** Let  $f$  and  $g$  be two distinct nonconstant meromorphic functions such that  $f$  and  $g$  share  $(0, k_1)$ ,  $(1, k_2)$  and  $(\infty, k_3)$ , where  $k_1, k_2$  and  $k_3$  are three positive integers satisfying (1.1). If  $N_0(r) \neq S(r, f)$ , then  $f$  and  $g$  share  $0, 1$  and  $\infty$  CM.

**Proof.** If  $f$  is a Möbius transformation of  $g$ , from Lemma 2.1 we can deduce the conclusion of Lemma 2.6. Next we suppose that  $f$  is not any Möbius transformation of  $g$ . First, from the condition that  $f$  and  $g$  share  $0, 1$  and  $\infty$  IM we deduce

$$S(r, f) = S(r, g). \quad (2.1)$$

Let

$$\frac{f-1}{g-1} = h_1, \quad (2.2)$$

$$\frac{f}{g} = h_2 \quad (2.3)$$

and

$$h_0 = \frac{h_1}{h_2}. \quad (2.4)$$

Then from (2.1)–(2.4) and Lemma 2.2 we get

$$\overline{N}(r, h_j) + \overline{N}(r, 1/h_j) = S(r, f) \quad (j = 0, 1, 2). \quad (2.5)$$

From (2.2)–(2.4) we deduce

$$f = \frac{h_1 - 1}{h_0 - 1} \quad (2.6)$$

and

$$g = \frac{h_1^{-1} - 1}{h_0^{-1} - 1}. \quad (2.7)$$

From (2.6) and (2.7) we deduce

$$f - g = \frac{(h_1 - 1)(1 - h_0 h_1^{-1})}{h_0 - 1}. \quad (2.8)$$

From (2.1) and (2.4)–(2.8) we deduce

$$N_0(r) = N_0(r, 1; h_1, h_0) + S(r, f) = N_0(r, 1; h_1, h_2) + S(r, f). \quad (2.9)$$

From (2.9) and the condition  $N_0(r) \neq S(r, f)$  we get

$$N_0(r, 1; h_1, h_2) \neq S(r, f). \quad (2.10)$$

From (2.5), (2.10) and Lemma 2.5 we see that there exist two integers  $s$  and  $t$  ( $|s| + |t| > 0$ ) such that

$$h_1^s h_2^t \equiv 1. \quad (2.11)$$

Substituting (2.2) and (2.3) into (2.11) we get

$$f^t (f - 1)^s \equiv g^t (g - 1)^s. \quad (2.12)$$

Noting that  $f$  is not any Möbius transformation of  $g$ , from (2.12) we deduce that  $s \neq 0$ ,  $t \neq 0$  and  $|s| \neq |t|$ . From this and (2.12) we deduce the conclusion of Lemma 2.6.  $\square$

**Lemma 2.7.** (See [18, Theorem 1].) Let  $f$  and  $g$  be two distinct nonconstant meromorphic functions such that  $f$  and  $g$  share 0, 1 and  $\infty$  CM. If

$$\limsup_{\substack{r \rightarrow \infty \\ r \notin E}} \frac{N_0(r)}{T(r, f)} > 1/2,$$

where  $E$  is a set of  $r$  of finite linear measure, then  $f$  is a Möbius transformation of  $g$ .

**Lemma 2.8.** (See [13, Lemma 6].) Let  $h_1$  and  $h_2$  be two distinct nonconstant meromorphic functions such that  $\overline{N}(r, h_j) + \overline{N}(r, 1/h_j) = S(r)$  ( $j = 1, 2$ ), where  $S(r) = o(T(r))$  ( $r \notin E$ ), as  $r \rightarrow \infty$ , and  $T(r) = \max\{T(r, h_1), T(r, h_2)\}$ , let  $a_0$  ( $\neq 0$ ),  $a_1$ ,  $a_2$  and  $a_3$  ( $\neq a_1, a_2$ ) be small functions of  $h_1$  and  $h_2$ , and let  $f = (a_0 + a_1 h_1 - a_2 h_2)/(h_1 - h_2)$ . If  $T(r, h_j) \neq S(r)$  ( $j = 1, 2$ ) and  $T(r, h_2/h_1) \neq S(r)$ , then  $T(r, f) = N(r, 1/(f - a_3)) + S(r)$ .

### 3. Proof of theorems

**Proof of Theorem 1.1.** We discuss the following two cases.

**Case 1.** Suppose that  $f$  is a Möbius transformation of  $g$ , then  $f$  and  $g$  assume one of the six relations (i)–(vi) in Lemma 2.1, and so it follows that  $f$  and  $g$  share 0, 1 and  $\infty$  CM. Thus from the condition  $\overline{E}_2(a, f) \subseteq \overline{E}_g(\{b\})$  and Theorem D we can see that  $f$  and  $g$  assume one of the nine relations (I)(i)–(ix) in Theorem D.

**Case 2.** Suppose that  $f$  is not any Möbius transformation of  $g$ , and let (2.2)–(2.4). Using proceeding as in the proof of Lemma 2.6 we deduce (2.1) and (2.5)–(2.9). We discuss the following two subcases.

**Subcase 2.1.** Suppose that

$$\overline{N}_{(2)}\left(r, \frac{1}{f-a}\right) - \overline{N}_{(3)}\left(r, \frac{1}{f-a}\right) \neq S(r, f). \quad (3.1)$$

Let

$$\varphi = \frac{f'(f-b)}{f(f-1)} - \frac{g'(g-b)}{g(g-1)}. \quad (3.2)$$

From (2.1), (3.2), Lemma 2.2 and the assumptions of Theorem 1.1 we deduce

$$T(r, \varphi) = S(r, f). \quad (3.3)$$

Noting that  $a \neq b$ , from (3.1), (3.3) and the condition  $\overline{E}_2(a, f) \subseteq \overline{E}_g(\{b\})$  we deduce  $\varphi \equiv 0$ , which reads

$$\frac{f'(f-b)}{f(f-1)} \equiv \frac{g'(g-b)}{g(g-1)}, \quad (3.4)$$

and (3.4) can be rewritten as

$$\frac{f'}{f-1} - \frac{g'}{g-1} \equiv \frac{b}{b-1} \cdot \left( \frac{f'}{f} - \frac{g'}{g} \right). \quad (3.5)$$

Let  $z_0$  be a zero of  $g-b$  with multiplicity  $\geq 2$ , but not a zero of  $f-b$  with multiplicity  $\geq 3$ , then it follows from (3.4) that  $f'(z_0) = g'(z_0) = 0$ . Combining (2.2), (2.5) and (ii) in Lemma 2.4 we deduce

$$\begin{aligned} N_{(2)}\left(r, \frac{1}{g-b}\right) &= 2\left(\overline{N}_{(2)}\left(r, \frac{1}{g-b}\right) - \overline{N}_{(3)}\left(r, \frac{1}{g-b}\right)\right) + N_{(3)}\left(r, \frac{1}{g-b}\right) \\ &\leq 2N\left(r, \frac{h_1}{h'_1}\right) + S(r, f) \leq 2T\left(r, \frac{h'_1}{h_1}\right) + S(r, f) = S(r, f), \end{aligned}$$

namely

$$N_{(2)}\left(r, \frac{1}{g-b}\right) = S(r, f). \quad (3.6)$$

Similarly, from (3.4) we deduce

$$N_{(2)}\left(r, \frac{1}{f-b}\right) = S(r, f). \quad (3.7)$$

Noting that  $a \neq b$  and the condition that  $f$  and  $g$  share 0, 1 and  $\infty$  CM, from (3.4) and the condition  $\overline{E}_2(a, f) \subseteq \overline{E}_g(\{b\})$  we deduce

$$N_{(1)}\left(r, \frac{1}{f-a}\right) = 0. \quad (3.8)$$

From (3.8) and Lemma 2.3 we can see that  $f$  and  $g$  are given by one of the nine expressions in Lemma 2.3. Suppose that  $f$  and  $g$  assume the form (i) of Lemma 2.3, then

$$f = \frac{e^{3\gamma} - 1}{e^\gamma - 1}, \quad f = \frac{e^{-3\gamma} - 1}{e^{-\gamma} - 1}, \quad (3.9)$$

with  $a = \frac{3}{4}$ . From (3.5)–(3.7) and (3.9) we can deduce  $b = 3$ , and so we obtain the conclusion (II)(i) of Theorem D. Suppose that  $f$  and  $g$  assume one of the forms (ii)–(ix) in Lemma 2.3, in the same manner as above we can obtain the conclusions (II)(ii)–(ix) of Theorem D, respectively.

**Subcase 2.2.** Suppose that

$$\bar{N}_{(2)}\left(r, \frac{1}{f-a}\right) - \bar{N}_{(3)}\left(r, \frac{1}{f-a}\right) = S(r, f). \quad (3.10)$$

Let

$$F = (f-a)(h_0-1) = h_1 - ah_0 + a - 1 \quad (3.11)$$

and

$$\omega = \frac{F'}{F}. \quad (3.12)$$

From (3.10) and (ii) and (iv) of Lemma 2.4 we deduce

$$T(r, f) = N_{(1)}\left(r, \frac{1}{f-a}\right) + S(r, f). \quad (3.13)$$

From (2.5), (3.11) and (3.13) we deduce that  $F$  is not a constant. If  $\omega$  is a constant, it follows from (3.12) that  $F = A_0 e^{B_0 z}$ , where  $A_0$  and  $B_0$  are nonzero constants. Let  $z_a$  be a simple zero of  $f-a$ , then from (3.11) we deduce that  $z_a$  be a pole of  $h_0$ , and so it follows from (2.5) that  $\bar{N}_{(1)}(r, 1/(f-a)) \leq \bar{N}(r, h_0) = S(r, f)$ , which contradicts (3.13). Thus  $\omega$  is not a constant. From (2.2)–(2.4) we deduce

$$\frac{f-g}{f(g-1)} = \frac{h_1-h_2}{h_2} = h_0 - 1. \quad (3.14)$$

Noting that  $a \neq b$ , from (3.14) and the condition  $\bar{E}_2(a, f) \subseteq \bar{E}_g(\{b\})$  we deduce that  $f-a$  and  $h_0-1$  have no common zero. From (2.5) and the second fundamental theorem we deduce  $T(r, h_0) = \bar{N}(r, 1/(h_0-1)) + S(r, f)$ , which implies that

$$N_{(2)}(r, 1/(h_0-1)) = S(r, f). \quad (3.15)$$

From (2.1), (2.9), (3.10)–(3.12), (3.14) and (ii) of Lemma 2.4 we deduce

$$T(r, \omega) = N(r, \omega) + S(r, f) = N_{(1)}\left(r, \frac{1}{F}\right) + S(r, f) = N_{(1)}\left(r, \frac{1}{f-a}\right) + N_0(r) + S(r, f). \quad (3.16)$$

Let

$$h = \frac{\alpha}{\alpha - \beta}, \quad (3.17)$$

where  $\alpha = h'_1/h_1$  and  $\beta = h'_2/h_2$ . Then it follows from (3.17) that

$$h'_0 = \frac{\alpha}{h} \cdot h_0 \quad (3.18)$$

and

$$h''_0 = \lambda h_0, \quad (3.19)$$

where

$$\lambda = \left(\frac{\alpha}{h}\right)' + \left(\frac{\alpha}{h}\right)^2. \quad (3.20)$$



From (2.1), (2.5) and (3.17) we deduce

$$T(r, h) + T(r, \alpha) = S(r, f). \quad (3.21)$$

Let  $z_0$  be a simple zero of  $f - a$  such that  $h_0(z_0) \neq 0, \infty$  and  $\alpha(z_0) \neq 0$ , then

$$g(z_0) = b, \quad h_1(z_0) = \frac{a-1}{b-1}, \quad h_0(z_0) = \frac{b}{a} \cdot \frac{a-1}{b-1}. \quad (3.22)$$

From (3.11), (3.18), (3.19), (3.22) and by using the Taylor expansion of  $F$  about  $z_0$  we deduce

$$F = \tau_1(z_0)(z - z_0) + \tau_2(z_0)(z - z_0)^2 + \tau_3(z_0)(z - z_0)^3 + O((z - z_0)^4), \quad (3.23)$$

where

$$\tau_1(z) = \left( \alpha - b \cdot \frac{h'_0}{h_0} \right) \cdot \frac{a-1}{b-1}, \quad \tau_2(z) = \left( \alpha^2 + \alpha' - b \cdot \frac{h''_0}{h'_0} \cdot \frac{\alpha}{h} \right) \cdot \frac{a-1}{2(b-1)}$$

and

$$\tau_3 = \frac{1}{2} \cdot \frac{a-1}{b-1} \cdot \alpha \alpha' + \frac{a-1}{6(b-1)} \cdot \alpha^3 + \frac{a-1}{6(b-1)} \cdot \alpha'' - \frac{a}{6} \cdot \left( \lambda' \cdot \frac{b}{a} \cdot \frac{a-1}{b-1} + \lambda \cdot \frac{\alpha}{h} \cdot \frac{b}{a} \cdot \frac{a-1}{b-1} \right).$$

From (2.5) and (3.21) we deduce

$$T(r, \tau_j) = S(r, f) \quad (j = 1, 2, 3). \quad (3.24)$$

If  $\tau_1 \equiv 0$ , then  $\alpha/b = h'_0/h_0$ , and so it follows from (3.18) that  $h \equiv b$ , from which and

$$\frac{h'_0}{h_0} \cdot (g - h) = \frac{f'}{f} \cdot \frac{g - f}{f - 1}$$

we get

$$\frac{h'_0}{h_0} \cdot (g - b) = \frac{f'}{f} \cdot \frac{g - f}{f - 1}. \quad (3.25)$$

Let  $z_1$  be a simple zero of  $f - a$  such that  $h'_0(z_1) \neq 0$  and  $1/h_0(z_1) \neq 0$ . Noting that  $a \neq b$ , from (3.25) and the condition  $\bar{E}_2(a, f) \subseteq \bar{E}_g(\{b\})$  we deduce  $f'(z_1) = 0$ , this is impossible. Thus  $\tau_1 \neq 0$ , and from (3.12) and (3.23) we deduce

$$\omega = \frac{1}{z - z_0} + \frac{B(z_0)}{2} + C(z_0) \cdot (z - z_0) + O((z - z_0)^2), \quad (3.26)$$

where  $B = \frac{2\tau_2}{\tau_1}$  and  $C = \frac{2\tau_3}{\tau_1} - (\frac{\tau_2}{\tau_1})^2$ . Let

$$H_* = \omega' + \omega^2 - B\omega - A, \quad (3.27)$$

where  $A = 3C - \frac{1}{4} \cdot B^2 - B'$ . From (3.26) and (3.27) we deduce  $H_*(z) = O(z - z_0)$ . Suppose that  $H_*(z) \not\equiv 0$ , then it follows from (3.12), (3.24) and (3.27) that

$$N_1\left(r, \frac{1}{f - a}\right) \leq N\left(r, \frac{1}{H_*}\right) \leq N(r, H_*) + S(r, f). \quad (3.28)$$

Let  $z_*$  be a simple zero of  $F$  such that  $f(z_*) \neq a$ . Then in the same manner as in the proof of (3.26) we have  $\omega = 1/(z - z_*) + B(z_*)/2 + O(z - z_*)$ . Combining (3.27) we deduce that  $z_*$  must be at most simple pole of  $H_*$ , and so it follows from (3.13), (3.16), (3.24) and (3.28) that

$$T(r, f) \leq N(r, H_*) + S(r, f) = N_0(r) + S(r, f). \quad (3.29)$$

From Lemmas 2.6 and 2.7 we deduce that  $f$  is a Möbius transformation of  $g$ , this contradicts the above supposition. Thus  $H_*(z) \equiv 0$ , which reads  $\omega' = A + B\omega - \omega^2$ , and so  $\omega'/\omega = A/\omega - \omega + B$ . Combining (3.12) we deduce  $F'' = AF + BF'$ , and so it follows from (2.1)–(2.4), (3.11) and (3.18)–(3.20) that

$$f(B_0f - B_0 - B_2) + g(f(B_1 + B_2) - B_1) \equiv 0, \quad (3.30)$$

where

$$B_0 = \alpha' + \alpha^2 - A - B\alpha, \quad (3.31)$$

$$B_1 = aA + aB \cdot \frac{\alpha}{h} - a \left( \left( \frac{\alpha}{h} \right)' + \left( \frac{\alpha}{h} \right)^2 \right) \quad (3.32)$$

and

$$B_2 = -(a - 1)A. \quad (3.33)$$

Suppose that  $f(B_1 + B_2) - B_1 \equiv 0$ , then it follows from (3.30) that  $B_0f - B_0 - B_2 \equiv 0$ . Combining (3.24), (3.26) and (3.27) we deduce that  $T(r, B_j) = S(r, f)$  ( $j = 0, 1, 2$ ), and so  $B_0 \equiv B_1 \equiv B_2 \equiv 0$ , which and (3.31)–(3.33) give

$$\alpha(h - 1) \equiv -h'. \quad (3.34)$$

Noting that  $f$  is not any Möbius transformation of  $g$ , from (2.3) and (3.17) we deduce  $h \neq 1$ , and so it follows from (3.34) that  $1/h_1 = A_*(h - 1)$ , where  $A_*$  is a certain constant. From this and (2.2) we get

$$g - 1 = A_*(h - 1)(f - 1). \quad (3.35)$$

From (3.13), (3.21), (3.35) and the condition  $\bar{E}_2(a, f) \subseteq \bar{E}_g(\{b\})$  we deduce  $b - 1 \equiv A_*(h - 1)(a - 1)$ , from this and (3.35) we can see that  $f$  is a Möbius transformation of  $g$ , this is a contradiction. Thus  $f(B_1 + B_2) - B_1 \not\equiv 0$ , and so it follows from (3.30) that

$$g = \frac{f(A_1f + A_2)}{A_3f + A_4}, \quad (3.36)$$

where  $A_1, A_2, A_3$  and  $A_4$  are small functions of  $f$  and  $g$ . Next we suppose that none of  $A_1, A_2, A_3$  and  $A_4$  is identically zero. If  $A_1A_4 - A_3A_2 \equiv 0$ , it follows from (3.36) that  $g = (A_2/A_4)f$ . Combining (3.13) and the condition  $\bar{E}_2(a, f) \subseteq \bar{E}_g(\{b\})$  we deduce  $A_2/A_4 \equiv b/a$ , and so  $f$  is a Möbius transformation of  $g$ , this is a contradiction. Thus  $A_1A_4 - A_3A_2 \not\equiv 0$ . Combining (3.36) and Lemma 2.2 we deduce

$$\bar{N}\left(r, \frac{1}{f + A_2/A_1}\right) + \bar{N}\left(r, \frac{1}{f + A_4/A_3}\right) = S(r, f). \quad (3.37)$$

From (3.37) and Nevanlinna's three small functions theorem (see [16, Theorem 1.36]) we deduce

$$T(r, f) = N_1\left(r, \frac{1}{f}\right) + S(r, f) = N_1(r, f) + S(r, f). \quad (3.38)$$

From (3.36) and (3.38) we deduce

$$m\left(r, \frac{1}{f}\right) + m(r, f) + N_2\left(r, \frac{1}{f}\right) + N_2(r, g) = S(r, f). \quad (3.39)$$

From (3.39) and Lemma 2.2 we deduce

$$N\left(r, \frac{g}{f}\right) \leq N_2\left(r, \frac{1}{f}\right) + N_2(r, g) + S(r, f) = S(r, f). \quad (3.40)$$

On the other hand, from (3.36) we deduce

$$T(r, g) = 2T(r, f) + S(r, f), \quad (3.41)$$

which and (2.1)–(2.4) give

$$T(r, h_j) \neq S(r, f) \quad (j = 0, 1, 2). \quad (3.42)$$

On the other hand, (2.6) can be rewritten as

$$f = \frac{1 - k_2}{k_1 - k_2}, \quad (3.43)$$

where

$$k_1 = h_2^{-1}, \quad k_2 = h_1^{-1}. \quad (3.44)$$

From (2.4), (3.42) and (3.44) we deduce

$$T(r, k_j) \neq S(r, f) \quad (j = 1, 2), \quad T\left(r, \frac{k_2}{k_1}\right) \neq S(r, f). \quad (3.45)$$

Suppose that  $A_4/A_3 \neq -1$ , and let  $a_0 = a_2 = 1$ ,  $a_1 = 0$  and  $a_3 = -A_4/A_3$ . Then from (3.43), (3.45) and Lemma 2.8 we get

$$T(r, f) = N\left(r, \frac{1}{A_3 f + A_4}\right) + S(r, f),$$

and so

$$m\left(r, \frac{1}{A_3 f + A_4}\right) = S(r, f). \quad (3.46)$$

From (2.3), (3.36), (3.39), (3.40) and (3.46) we deduce

$$T(r, h_2) = m\left(r, \frac{g}{f}\right) + N\left(r, \frac{g}{f}\right) + O(1) \leq m(r, f) + m\left(r, \frac{1}{A_3 f + A_4}\right) + S(r, f) = S(r, f), \quad (3.47)$$

which contradicts (3.42). Thus  $A_4/A_3 \equiv -1$ . Combining the condition  $A_1 A_4 - A_3 A_2 \neq 0$  we deduce  $A_2/A_1 \neq -1$ . From this, (3.36) and the condition that  $f$  and  $g$  share 1 IM we deduce

$$N\left(r, \frac{1}{f-1}\right) = S(r, f). \quad (3.48)$$

From (3.41), (3.48) and (v) of Lemma 2.4 we deduce

$$T(r, f) = N_0(r) + S(r, f). \quad (3.49)$$

From (3.49), Lemmas 2.6 and 2.7 we deduce that  $f$  is a Möbius transformation of  $g$ , this is a contradiction. Thus  $A_2/A_1 \equiv -1$ . Combining the condition  $A_4/A_3 \equiv -1$  we have  $A_1 A_4 - A_3 A_2 \equiv 0$ , this is impossible. Thus, at least one of  $A_1$ ,  $A_2$ ,  $A_3$  and  $A_4$  is identically zero. We discuss the following four subcases.

**Subcase 2.2.1.** Suppose that  $A_1 \equiv 0$ . If  $A_3 \equiv 0$ , then (3.36) can be rewritten as  $g = (A_2/A_4)f$ . Combining (3.13) and the condition  $\bar{E}_2(a, f) \subseteq \bar{E}_g(\{b\})$  we deduce  $A_2/A_4 \equiv b/a$ , and so  $f$  is a Möbius transformation of  $g$ , this is a contradiction. Thus  $A_3 \neq 0$ , and so (3.36) can be rewritten as

$$g = \frac{A_2}{A_3} \cdot \frac{f}{f + A_4/A_3}. \quad (3.50)$$

If  $-A_4/A_3 \equiv 1$ , from (3.13) and the condition  $\bar{E}_2(a, f) \subseteq \bar{E}_g(\{b\})$ , we deduce  $A_2/A_3 \equiv (ba - b)/a$ , and so it follows from (3.50) that  $f$  is a Möbius transformation of  $g$ , this is a contradiction. Thus  $-A_4/A_3 \neq 1$ . Combining (3.50) and Nevanlinna's three small functions theorem (see [16, Theorem 1.36]) we deduce

$$T(r, f) \leq \bar{N}\left(r, \frac{1}{f-1}\right) + \bar{N}\left(r, \frac{1}{f + A_4/A_3}\right) + \bar{N}(r, f) + S(r, f) = \bar{N}\left(r, \frac{1}{f-1}\right) + S(r, f),$$

which implies that  $T(r, f) = \bar{N}(r, 1/(f-1)) + S(r, f)$ , and so it follows from (3.13) and the condition  $\bar{E}_2(a, f) \subseteq \bar{E}_g(\{b\})$  we deduce

$$\frac{A_2}{A_3} \cdot \frac{1}{1 + A_4/A_3} \equiv 1 \quad \text{and} \quad \frac{A_2}{A_3} \cdot \frac{a}{a + A_4/A_3} \equiv b. \quad (3.51)$$

From (3.51) we deduce that  $A_2/A_3$  and  $A_4/A_3$  are constants, and so it follows from (3.50) that  $f$  is a Möbius transformation of  $g$ , this is a contradiction. Thus  $A_1 \neq 0$ .

**Subcase 2.2.2.** Suppose that  $A_2 \equiv 0$  and  $A_1 \neq 0$ . Then (3.36) can be rewritten as

$$g = \frac{A_1 f^2}{A_3 f + A_4}. \quad (3.52)$$

If  $A_4 \equiv 0$ , then it follows from (3.52) that  $g = (A_1/A_3)f$ . Combining (3.13) and the condition  $\overline{E}_2(a, f) \subseteq \overline{E}_g(\{b\})$  we deduce  $A_1/A_3 \equiv b/a$ , and so  $f$  is a Möbius transformation of  $g$ , this is a contradiction. Thus  $A_4 \neq 0$ . If  $A_3 \equiv 0$ , then it follows from (3.52) that  $g = (A_1/A_4)f^2$ . Combining Lemma 2.2 and the second fundamental theorem we get

$$\begin{aligned} 2T(r, f) &= T(r, g) + S(r, f) \leq N_1\left(r, \frac{1}{g}\right) + N_1(r, g) + \overline{N}\left(r, \frac{1}{g-1}\right) + S(r, f) \\ &= \overline{N}\left(r, \frac{1}{g-1}\right) + S(r, f) \leq T(r, f) + S(r, f), \end{aligned}$$

which implies that  $T(r, f) = S(r, f)$ , this is impossible. Thus  $A_3 \neq 0$ , and so it follows from (3.52) that

$$g = \frac{A_1}{A_3} \cdot \frac{f^2}{f + A_4/A_3}. \quad (3.53)$$

If  $A_4/A_3 \neq -1$ , from (3.53), Lemma 2.2 and Nevanlinna's three small functions theorem (see [16, Theorem 1.36]) we get

$$\begin{aligned} 2T(r, f) &\leq \overline{N}\left(r, \frac{1}{f + A_4/A_3}\right) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) + S(r, f) \\ &= \overline{N}\left(r, \frac{1}{f-1}\right) + S(r, f) \leq T(r, f) + S(r, f), \end{aligned}$$

which implies that  $T(r, f) = S(r, f)$ , this is impossible. Thus  $A_4/A_3 \equiv -1$ , and so it follows from (3.53) that

$$\begin{aligned} 2T(r, f) + S(r, f) &= T(r, g) \leq \overline{N}\left(r, \frac{1}{g-1}\right) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r, g) + S(r, f) \\ &= \overline{N}(r, g) + S(r, f) \leq T(r, f) + S(r, f), \end{aligned}$$

which implies that  $T(r, f) = S(r, f)$ , this is impossible. Thus  $A_2 \neq 0$ .

**Subcase 2.2.3.** Suppose that  $A_3 \equiv 0$  and  $A_1 A_2 A_4 \neq 0$ . Then (3.36) can be rewritten as

$$g = (A_1/A_4) \cdot f(f + A_2/A_1). \quad (3.54)$$

From (3.54) and Lemma 2.2 we deduce  $T(r, g) = 2T(r, f) + S(r, f)$  and  $\overline{N}(r, f) = S(r, f)$ . From this, (v) in Lemma 2.4 and the condition that  $f$  is not any Möbius transformation of  $g$ , we deduce (3.49). From (3.49), Lemmas 2.6 and 2.7 we deduce that  $f$  is a Möbius transformation of  $g$ , this is a contradiction.

**Subcase 2.2.4.** Suppose that  $A_4 \equiv 0$  and  $A_1 A_2 A_3 \neq 0$ . Then (3.36) can be rewritten as

$$g = (A_1/A_3) \cdot (f + A_2/A_1). \quad (3.55)$$

If  $A_2/A_1 \neq -1$ , from (3.55), Lemma 2.2 and Nevanlinna's three small functions theorem (see [16, Theorem 1.36]) we deduce

$$T(r, f) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f + A_2/A_1}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) + S(r, f) = \overline{N}\left(r, \frac{1}{f-1}\right) + S(r, f),$$

which implies that  $T(r, f) = N_1(r, \frac{1}{f-1}) + S(r, f)$ . From this, (3.13), (3.55) and the condition  $\overline{E}_2(a, f) \subseteq \overline{E}_g(\{b\})$  we deduce that  $A_1/A_3$  and  $A_2/A_1$  are constants, and so it follows from (3.55) that  $f$  is a Möbius transformation of  $g$ , this is a contradiction. Thus  $A_2/A_1 \equiv -1$ , and it follows from (3.13) and (3.55) that  $A_1/A_3$  is a constant, and so from (3.55) we can see that  $f$  is a Möbius transformation of  $g$ , this is a contradiction.

Theorem 1.1 is thus completely proved.  $\square$

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